Module–III:
Sequence and Series

CHAPTER 9

Sequences

Sequences and Series

(Convergence of Sequences and series, Tests for convergence of series, Ratio test, D'Alembert's test, Raabe's Test)

9.1 INTRODUCTION

A series is the sum of the terms of an infinite sequence of numbers. Given an infinite sequence, the n^{th} partial sum is the sum of the first n terms of the sequence. A series is convergent if the sequence of its partial sums S_1, S_2, S_3, ... tends to a limit, partial sums become closer and closer to a given number when the number of their terms increases.

A series converges, if there exists a number such that for any arbitrarily small positive number, there is sufficiently large integer N... In convergent series the number (necessarily unique) is called the sum of the series.

Convergence and divergence are unaffected by deleting a finite number of terms from the beginning of a series. Any series that is not convergent is said to be divergent.

9.2 SEQUENCE

A *sequence* is a succession of numbers or terms formed according to some definite rule. The n^{th} term in a sequence is denoted by u_n.

For example, if $u_n = 2n + 1$.

By giving different values of n in u_n, we get different terms of the sequence.

Thus, $u_1 = 3$, $u_2 = 5$, $u_3 = 7$, ...

A sequence having unlimited number of terms is known as an *infinite sequence.*

9.3 LIMIT

If a sequence tends to a limit l, then we write $\lim\limits_{n\to\infty}(u_n) = l$

9.4 INCREASING AND DECREASING SEQUENCES

(*i*) **Increasing sequence:** The sequence $\sum\limits_{n=0}^{m} a_n$ is said to be increasing if $a_{n+1} \geq a_n$, for each $n \leq m$.

(*ii*) **Decreasing sequence:** The sequence $\sum\limits_{n=0}^{m} a_n$ is said to be decreasing if $a_n \geq a_{n+1}$, for each $n \leq m$.

Example 1. *Show that the sequence* $\left\{\frac{3}{n+3}\right\}$ *is a decreasing sequence.* (GTU, June 201...)

Solution. We know that, the sequence $\sum_{n=0}^{m} a_n$ is decreasing if $a_n \geq a_{n+1}$ for each $n \leq m$.

Here we have
$$a_n = \frac{3}{n+3}$$

And
$$a_{n+1} = \frac{3}{n+1+3} = \frac{3}{n+4} < \frac{3}{n+3}$$

Thus we observe that $a_n \geq a_{n+1}$

Hence the sequence $\left\{\frac{3}{n+3}\right\}$ is a decreasing sequence.

Proved

9.5 CONVERGENT SEQUENCE (V.Imp)

If the limit of a sequence is finite, the sequence is ***convergent***. If the limit of a sequence does not tend to a finite number, the sequence is said to be ***divergent***.

e.g., $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots, \frac{1}{n^2} + \ldots$ is a convergent sequence.

$3, 5, 7, \ldots, (2n+1), \ldots$ is a divergent sequence.

9.6 BOUNDED SEQUENCE (V.Imp)

$u_1, u_2, u_3 \ldots, u_n \ldots$ is a bounded sequence if $u_n < k$ for every n.

9.7 MONOTONIC SEQUENCE (V.Imp)

The sequence is either increasing or decreasing, such sequences are called *monotonic*.

e.g., $1, 4, 7, 10, \ldots$ is a monotonic sequence. Increasing

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is also a monotonic sequence. Decreasing

$1, -1, 1, -1, 1, \ldots$ is not a monotonic sequence.

A sequence which is monotonic and bounded is a convergent sequence.

EXERCISE 9.1

Determine the general term of each of the following sequence. Prove that the following sequences are convergent or divergent.

1. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ **Ans.** $\frac{1}{2^n}$
2. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ **Ans.** $\frac{n}{n+1}$
3. $1, -1, 1, -1, \ldots$ **Ans.** $(-1)^{n-1}$
4. $\frac{1^2}{1!}, \frac{2^2}{2!}, \frac{3^2}{3!}, \frac{4^2}{4!}, \frac{5^2}{5!}, \ldots$ **Ans.** $\frac{n^2}{n!}$

Which of the following sequences are convergent?

5. $u_n = \frac{n+1}{n}$ **Ans.** Convergent
6. $u_n = 3n$ **Ans.** Divergent
7. $u_n = n^2$ **Ans.** Divergent
8. $u_n = \frac{1}{n}$ **Ans.** Convergent

9.8 REMEMBER THE FOLLOWING LIMITS

(i) $\lim_{n\to\infty} x^n = 0$ if $x < 1$ and $\lim_{n\to\infty} x^n = \infty$ if $x > 1$

(ii) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all values of x (iii) $\lim_{n\to\infty} \frac{\log n}{n} = 0$

(iv) $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$ (v) $\lim_{n\to\infty} (n)^{1/n} = 1$

(vi) $\lim_{n\to\infty} (n)^{1/n} = \infty$ (vii) $\lim_{n\to\infty} \left[\frac{(n!)}{n}\right]^{1/n} = \frac{1}{e}$

(viii) $\lim_{n\to\infty} n\, x^n = 0$ if $x < 1$ (ix) $\lim_{n\to\infty} n^h = \infty$

(x) $\lim_{n\to\infty} \frac{1}{n^h} = 0$ (xi) $\lim_{x\to\infty} \left[\frac{a^x - 1}{x}\right] = \log a$ or $\lim_{n\to\infty} \frac{a^{1/n} - 1}{1/n} = \log a$

(xii) $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (xiii) $\lim_{x\to 0} \frac{\tan x}{x} = 1$

9.9 SERIES

A *series* is the sum of a sequence.

Let $u_1, u_2, u_3, \ldots\ldots\ldots, u_n, \ldots\ldots\ldots$ be a given sequence. Then, the expression $u_1 + u_2 + u_3 + \ldots\ldots\ldots + u_n + \ldots\ldots\ldots$ is called the series associated with the given sequence.

For example, $1 + 3 + 5 + 7 + \ldots$ is a series.

If the number of terms of a series is limited, the series is called *finite*. When the number of terms of a series are unlimited, it is called an *infinite series*.

$$u_1 + u_2 + u_3 + u_4 + \ldots + u_n + \ldots \infty$$

is called an infinite series and it is denoted by $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$. The sum of the first n terms of a series is denoted by S_n.

9.10 CONVERGENT, DIVERGENT AND OSCILLATORY SERIES

Consider the infinite series $\Sigma u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n$$

Three cases arise:

(i) If S_n tends to a finite number as $n \to \infty$, the series Σu_n is said to be *convergent*.

(ii) If S_n tends to infinity as $n \to \infty$, the series Σu_n is said to be *divergent*.

(iii) If S_n does not tend to a unique limit, finite or infinite, the series Σu_n is called *oscillatory*.

9.11 PROPERTIES OF SERIES

1. The nature of an infinite series does not change:
   (i) by multiplication of all terms by a constant k.
   (ii) by addition or deletion of a finite number of terms.
2. If two series Σu_n and Σv_n are convergent, then $\Sigma(u_n + v_n)$ is also convergent.

V Imp

Example 2. *Examine the nature of the series* $1 + 2 + 3 + 4 + \ldots + n + \ldots$

Solution. Let
$$S_n = 1+2+3+4+\ldots+n = \frac{n(n+1)}{2} \quad \text{[Series in A.P.]}$$

Since
$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} \Rightarrow \infty$$

Hence, this series is divergent.

Example 3. *Test the convergence of the series* $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\infty$ (V Imp)

Solution. Let
$$S_n = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\infty \quad \text{[Series in G.P.]}$$
$$= \frac{1}{1-\frac{1}{2}} = 2 \qquad \left(S_n = \frac{a}{1-r}\right)$$
$$\lim_{n\to\infty} S_n = 2$$

$1+\frac{1}{n}$

$1+\frac{1}{1}$

$1+4+7+\ldots$

$\frac{n(n+3)}{2}$

Hence, the series is convergent. Ans.

Example 4. *Prove that the following series:* $\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots$ *is convergent and find its sum.* (M.U. 2008)

Solution. Here,
$$u_n = \frac{n+1}{(n+2)!} = \frac{n+2-1}{(n+2)!} = \frac{n+2}{(n+2)!} - \frac{1}{(n+2)!}$$
$$= \frac{1}{(n+1)!} - \frac{1}{(n+2)!}$$

$r = \frac{2\text{nd term}}{1\text{st term}}$

$$S_n = \left(\frac{1}{2!}-\frac{1}{3!}\right)+\left(\frac{1}{3!}-\frac{1}{4!}\right)+\left(\frac{1}{4!}-\frac{1}{5!}\right)+\ldots$$
$$+\left(\frac{1}{(n+1)!}-\frac{1}{(n+2)!}\right) = \frac{1}{2!}-\frac{1}{(n+2)!}$$
$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{2!}-\frac{1}{(n+2)!}\right] = \frac{1}{2}$$

$\therefore$ Σu_n converges and its limit is $\frac{1}{2}$. Ans.

Example 5. *Discuss the nature of the series* $2-2+2-2+2-\ldots$

Solution. Let
$$S_n = 2-2+2-2+2-\ldots\infty$$
$$= 0 \text{ if } n \text{ is even}$$
$$= 2 \text{ if } n \text{ is odd.}$$

Hence, S_n does not tend to a unique limit, and, therefore, the given series is oscillatory. Ans.

EXERCISE 9.2

Discuss the nature of the following series:

1. $1+4+7+10+\ldots\infty$ — **Ans.** Divergent
2. $1+\frac{5}{4}+\frac{6}{4}+\frac{7}{4}+\ldots\infty$ — **Ans.** Divergent
3. $6-5-1+6-5-1+6-5-1+\ldots\infty$ — **Ans.** Oscillatory
4. $3+\frac{3}{2}+\frac{3}{2^2}+\infty$ — **Ans.** Convergent
5. $1^2+2^2+3^2+4^2+\ldots\infty$ — **Ans.** Divergent
6. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots\infty$ — **Ans.** Convergent
7. $\frac{1}{1.3}+\frac{1}{1.3}+\frac{1}{5.7}+\ldots\infty$ — **Ans.** Convergent
8. $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots\infty$ — **Ans.** Convergent
9. $\log 3+\log\frac{4}{3}+\log\frac{5}{4}+\ldots\infty$ — **Ans.** Divergent
10. $\sum\log\frac{n}{n+1}$ — **Ans.** Divergent
11. $\Sigma\left(\sqrt{n+1}-\sqrt{n}\right)$ — **Ans.** Divergent
12. $\sum\frac{1}{n(n+2)}$ — **Ans.** Convergent
13. $\sum\frac{1}{n(n+1)(n+2)(n+3)}$ — **Ans.** Convergent
14. $\sum\frac{n}{(n+1)(n+2)(n+3)}$ — **Ans.** Convergent
15. $\sum\frac{2n+1}{n^2(n+1)^2}$ — **Ans.** Convergent

Covergent if $|r|<1$

Divergent if $r \geq 1$

oscillating if $r \leq -1$

9.12 PROPERTIES OF GEOMETRIC SERIES

The series $1+r+r^2+r^3+\ldots\infty$ is

(i) convergent if $|r|<1$ (ii) divergent if $r \geq 1$. (iii) oscillatory if $r \leq -1$.

Proof. $S_n = 1 + r + r^2 + ... + r^{n-1} = \dfrac{1-r^n}{1-r}$

(*i*) When $|r| < 1$, $\lim_{n\to\infty} r^n = 0$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{1-0}{1-r} = \frac{1}{1-r}$$

Hence, the series is convergent.

(*ii*) (*a*) When $r > 1$, $\lim_{n\to\infty} r^n = \infty$ $\quad \therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{r^n - 1}{r-1} = \infty$

Hence, the series is divergent.

(*b*) When $r = 1$, the series becomes $1 + 1 + 1 + 1 + ... \infty$

$$S_n = 1 + 1 + 1 + 1 + ... = n$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$$

Hence, the series is divergent.

(*iii*) (*a*) When $r = -1$, the series becomes $1 - 1 + 1 - 1 + 1 - ... \infty$.

$S_n = 0$ if n is even

$= 1$ if n is odd

Hence, the series is oscillatory.

(*b*) When $r < -1$, let $r = -k$ where $k > 1$.

$$r^n = (-k)^n = (-1)^n k^n$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-r^n}{1-r} = \lim_{n\to\infty} \frac{1-(-1)^n k^n}{1-(-k)}$$

$= +\infty$ if n is odd

$= -\infty$ if n is even

Hence, the series is oscillatory.

Proved.

EXERCISE 9.3

Test the nature of the following series:

1. $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ...\infty$ **Ans.** Convergent
2. $1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + ...\infty$ **Ans.** Convergent
3. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + ...\infty$ **Ans.** Convergent
4. $1 - 2 + 4 - 8 + ... \infty$ **Ans.** Oscillatory
5. $2 + 3 + \frac{9}{2} + \frac{27}{4} + ...\infty$ **Ans.** Divergent
6. $1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^3 + ...\infty$ **Ans.** Divergent

7. State, which one of the alternatives in the following is correct:
The series $1 - 1 + 1 - 1 + ...$ is
(*a*) Convergent with its sum equal to 0.
(*b*) Convergent with its sum equal to 1.
(*c*) Divergent.
(*d*) Oscillatory.
Ans. (*d*)

8. The sum of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is [*G.T.U, June, December 2015*]
(*a*) 0 (*b*) 1 (*c*) −1 (*d*) 1/2
Ans. (*b*)

9. Define the Geometric series and find the sum of the following series $\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}}$
[*G.T.U, June 2015*]

9.13 POSITIVE TERM SERIES

If all terms after few negative terms in an infinite series are positive, such a series is a positive term series.

e.g., $-10 - 6 - 1 + 5 + 12 + 20 + ...$ is a positive term series.

By omitting the negative terms, the nature of a positive term series remains unchanged.

9.14 NECESSARY CONDITIONS FOR CONVERGENT SERIES

For every convergent series Σu_n,

$$\lim_{n\to\infty} u_n = 0$$

Solution. Let $S_n = u_1 + u_2 + u_3 + ... + u_n$

$\lim_{n\to\infty} S_n = k$ (a finite quantity)

Also $\lim_{n\to\infty} S_{n-1} = k$ (a finite quantity)

$$S_n = S_{n-1} + u_n$$

$$u_n = S_n - S_{n-1}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}[S_n - S_{n-1}] = 0$$

$$\lim_{n\to\infty} u_n = 0$$

Corollary. Converse of the above theorem is not true.

e.g., $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + ... + \frac{1}{\sqrt{n}} + ...\infty$ is divergent.

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + ... + \frac{1}{\sqrt{n}}$$

$$> \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + ... + \frac{1}{\sqrt{n}}$$

$$> \frac{n}{\sqrt{n}} > \sqrt{n}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{n} = \infty$$

Thus, the series is divergent, although $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

So $\lim_{n\to\infty} u_n = 0$ is a necessary condition but not a sufficient condition for convergence.

Note: 1. Test for divergence

If $\lim_{n\to\infty} u_n \neq 0$, the series Σu_n must be divergent.

2. *To determine the nature of a series we have to find S_n. Since it is not possible to find S_n for every series, we have to device tests for convergence without involving S_n.*

9.15 CAUCHY'S FUNDAMENTAL TEST FOR DIVERGENCE (The n^{th} term test for divergence)

If $\lim_{n\to\infty} u_n \neq 0$ the series is divergent.

Example 6. *Test for convergence of the series* $1+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\ldots+\frac{n}{n+1}+\ldots\infty$

Solution. Here,
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1 \neq 0$$

Hence, by **Cauchy's Fundamental Test** for divergence, the series is divergent. **Ans.**

Example 7. *Test for convergence the series* $1+\frac{3}{5}+\frac{8}{10}+\frac{15}{17}+\ldots+\frac{2^n-1}{2^n+1}+\ldots\infty$

Solution. Here,
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{2^n-1}{2^n+1} = \lim_{n\to\infty}\frac{1-\frac{1}{2^n}}{1+\frac{1}{2^n}} = 1 \neq 0$$

Hence, by **Cauchy's Fundamental Test** for divergence the series is divergent. **Ans.**

Example 8. *Test the convergence of the following series:*
$$\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\sqrt{\frac{3}{8}}+\ldots+\sqrt{\frac{n}{2(n+1)}}+\ldots$$

Solution. Here, we have
$$\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\sqrt{\frac{3}{8}}+\ldots+\sqrt{\frac{n}{2(n+1)}}+\ldots$$
$$u_n = \sqrt{\frac{n}{2(n+1)}} = \sqrt{\frac{1}{2\left(1+\frac{1}{n}\right)}}$$
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{2\left(1+\frac{1}{n}\right)}} = \frac{1}{\sqrt{2}} \neq 0$$

$\Rightarrow \Sigma u_n$ does not converge.

The given series is a series of +ve terms.
Hence by Cauchy fundamental test for divergence, the series is divergent. **Ans.**

Example 9. *Test for convergence the series whose n^{th} term is* $\left(1+\frac{1}{\sqrt{n}}\right)$ (G.T.U., Dec. 2013)

Solution. Here, we have
$$u_n = \left(1+\frac{1}{\sqrt{n}}\right)$$
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n}}\right)$$

$\Rightarrow$ Hence by *Cauchy's Fundamental Test* for divergence, Σu_n is divergent. **Ans.**

EXERCISE 9.4

Examine for convergence

1. $\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{5}}+\frac{4}{\sqrt{17}}+\ldots+\frac{2^n}{\sqrt{4^n+1}}+\ldots\infty$ **Ans. Divergent**
2. $\sum_{n=1}^{\infty}\sqrt{\frac{n}{n+1}}$ **Ans. Divergent**
3. $\sum\cos\frac{1}{n}$ **Ans. Divergent**
4. $1+\frac{1}{2}+2+\frac{1}{3}+3+\frac{1}{4}+4+\ldots$ **Ans. Divergent**
5. $\Sigma(6-n^2)$ **Ans. Divergent**
6. $\Sigma(-2^n)$ **Ans. Divergent**
7. $\Sigma 3^{n+1}$ **Ans. Divergent**

9.16 P SERIES

The series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots\infty$ *is (i) convergent if $p > 1$ (ii) Divergent if $p \leq 1$.* (*MDU, Dec. 2010*)

Solution. (*GTU, June. 2015*)

Case 1: $(p > 1)$

The given series can be grouped as

$p>1$ $p \leq 1$

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\left(\frac{1}{8^p}+\frac{1}{9^p}+\frac{1}{10^p}+\frac{1}{11^p}+\frac{1}{12^p}+\frac{1}{13^p}+\frac{1}{14^p}+\frac{1}{15^p}\right)+\ldots \quad \ldots(1)$$

Now
$$\frac{1}{1^p} = 1 \quad \ldots(2)$$
$$\frac{1}{2^p}+\frac{1}{3^p} < \frac{1}{2^p}+\frac{1}{2^p} = \frac{2}{2^p} \quad \ldots(3)$$
$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p} < \frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p} = \frac{4}{4^p} \quad \ldots(4)$$
$$\frac{1}{8^p}+\frac{1}{9^p}+\ldots+\frac{1}{15^p} < \frac{1}{8^p}+\frac{1}{8^p}+\ldots+\frac{1}{8^p} = \frac{8}{8^p}$$

On adding (1), (2), (3) and (4), we get:

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\left(\frac{1}{8^p}+\frac{1}{9^p}+\ldots+\frac{1}{15^p}\right)+\ldots$$

$$<\frac{1}{1^p}+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\ldots$$

$$<1+\left(\frac{1}{2}\right)^{p-1}+\left(\frac{1}{2}\right)^{2p-2}+\left(\frac{1}{2}\right)^{3p-3}+\ldots$$

$$<\frac{1}{1-\left(\frac{1}{2}\right)^{p-1}} \qquad \left[\text{G.P., } r=\left(\frac{1}{2}\right)^{p-1}, S=\frac{1}{1-r}\right]$$

$$<\text{Finite number if } p>1$$

Hence, the given series is convergent when $p > 1$.

Case 2: $p = 1$

When $p = 1$, the given series becomes

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\ldots+\frac{1}{16}\right)+\ldots$$

$$1+\frac{1}{2}=1+\frac{1}{2} \qquad \ldots(1)$$

$$\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{1}{2} \qquad \ldots(2)$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2} \qquad \ldots(3)$$

$$\frac{1}{9}+\frac{1}{10}+\ldots+\frac{1}{16}>\frac{1}{16}+\frac{1}{16}+\ldots+\frac{1}{16}=\frac{8}{16}=\frac{1}{2} \qquad \ldots(4)$$

On adding (1), (2), (3) and (4), we get

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\ldots+\frac{1}{16}\right)+\ldots a$$

$$>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$$

$$>1+\frac{n}{2} \quad (n\to\infty)$$

$$>\infty$$

Hence, the given series is divergent when $p = 1$.

Case 3: $p < 1$

$$\frac{1}{2^p}>\frac{1}{2}, \qquad \frac{1}{3^p}>\frac{1}{3}, \qquad \frac{1}{4^p}>\frac{1}{4} \text{ and so on}$$

Therefore, $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots>1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$

$>$ divergent series ($p = 1$). [From Case 2]

[As the series on R.H.S. $\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\right)$ is divergent]

Hence, the given series is divergent when $p < 1$.

9.17 COMPARISON TEST

Imp

If two positive terms Σu_n and Σv_n be such that

$\lim\limits_{n\to\infty}\frac{u_n}{v_n}=k$ **(finite number), then both series converge or diverge together.**

Proof. By definition of limit there exists a positive number ε, however small, such that

$$\left|\frac{u_n}{v_n}-k\right|<\varepsilon \text{ for } n>m \qquad i.e., -\varepsilon<\frac{u_n}{v_n}-k<+\varepsilon$$

$$k-\varepsilon<\frac{u_n}{v_n}<k+\varepsilon \text{ for } n>m$$

Ignoring the first m terms of both series, we have

$$k-\varepsilon<\frac{u_n}{v_n}<k+\varepsilon \text{ for all } n \qquad \ldots(1)$$

Case 1. Σv_n is convergent, then

$$\lim_{n\to\infty}(v_1+v_2+\ldots v_n)=h \text{ (say)} \qquad \text{where } h \text{ is a finite number.}$$

lim (1), $u_n < (k + \varepsilon)\, v_n$ for all n.

$$\lim_{n\to\infty}(u_1+u_2+\ldots+u_n)<(k+\varepsilon)\lim_{n\to\infty}(v_1+v_2+\ldots+v_n)=(k+\varepsilon)h$$

Hence, Σu_n is also convergent.

Case 2. Σv_n is divergent, then

$$\lim_{n\to\infty}(v_1+v_2+\ldots+v_n)\to\infty \qquad \ldots(2)$$

Now from (1) $\qquad k-\varepsilon<\frac{u_n}{v_n}$

$$u_n>(k-\varepsilon)v_n \text{ for all } n$$

$$\lim_{n\to\infty}(u_1+u_2+\ldots+u_n)>(k-\varepsilon)\lim_{n\to\infty}(v_1+v_2+\ldots+v_n)$$

From (2), $\lim\limits_{n\to\infty}(u_1+u_2+\ldots+u_n)\to\infty$

Hence Σu_n is also divergent.

Note: For testing the convergence of a series, this Comparison Test is very useful. We choose Σv_n (p-series) in such a way that

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\text{finite number.}$$

Then the nature of both the series is the same. The nature of Σv_n (p-series) is already known, so the nature of Σu_n is also known.

Example 10. *Test the series* $\sum_{n=1}^{\infty} \frac{1}{n+10}$ *for convergence or divergence.*

Solution. Here,
$$u_n = \frac{1}{n+10}$$

Let
$$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{n+10} = \lim_{n\to\infty} \frac{1}{1+\frac{10}{n}} = 1 = \text{finite number}$$

According to Comparison Test both series converge or diverge together, but Σv_n is divergent as $p = 1$.

$\therefore$ Σu_n is also divergent. **Ans.**

Example 11. *Test the convergence of the following series:* (V.Imp)

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \ldots$$

Solution. Here, we have

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \ldots$$

$$u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}\left[1+\sqrt{1+\frac{1}{n}}\right]}$$

Let us compare Σu_n with Σv_n, where

$$v_n = \frac{1}{\sqrt{n}}$$

$\frac{1}{\sqrt{2}}$ $\frac{1}{2} < 1$

$$\frac{u_n}{v_n} = \frac{1}{\sqrt{n}\left[1+\sqrt{1+\frac{1}{n}}\right]} \cdot \frac{\sqrt{n}}{1} = \frac{1}{1+\sqrt{1+\frac{1}{n}}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{1+1} = \frac{1}{2}$$

Which is finite and non-zero.

$\therefore$ Σv_n and Σv_n, converge or diverge together since $\Sigma v_n = \Sigma \frac{1}{n^{\frac{1}{2}}}$ is of the form $\Sigma \frac{1}{n^p}$,

$$p = \frac{1}{2} < 1$$

$\frac{1}{2} < 1$

$\therefore$ Σv_n is divergent $\Rightarrow$ Σu_n is also divergent. **Ans.**

$p < 1$

Div

Example 12. *Examine the convergence of the series:* $\Sigma\left(\sqrt[3]{n^3+1}-n\right)$

Solution. Here, we have $\Sigma\left(\sqrt[3]{n^3+1}-n\right)$

$$u_n = \left(n^3+1\right)^{\frac{1}{3}} - n = \left[n^3\left(1+\frac{1}{n^3}\right)\right]^{\frac{1}{3}} - n$$

$$= n\left[\left(1+\frac{1}{n^3}\right)^{\frac{1}{3}} - 1\right] = n\left[1+\frac{1}{3}\cdot\frac{1}{n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{n^3}\right)^2 + \ldots - 1\right]$$

$$= \frac{n}{n^3}\left[\frac{1}{3} - \frac{1}{9}\cdot\frac{1}{n^3}\ldots\right] = \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9}\cdot\frac{1}{n^3}\ldots\right]$$

Let
$$v_n = \frac{1}{n^2}$$

$$\frac{u_n}{v_n} = \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9}\cdot\frac{1}{n^3}\ldots\right]\cdot\frac{n^2}{1} = \left[\frac{1}{3} - \frac{1}{9}\cdot\frac{1}{n^3}\ldots\right]$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{9}\cdot\frac{1}{n^3}\ldots\right) = \frac{1}{3}$$

which is finite and non-zero.

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n^2}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = 2 > 1$

$\therefore$ Σv_n is convergent $\Rightarrow$ Σu_n is convergent. **Ans.**

Example 13. *Test the convergence of the following series* $\frac{1}{1+2^{-1}} + \frac{2}{1+2^{-2}} + \frac{3}{1+2^{-3}} + \ldots$

Solution. Here, we have

$$\frac{1}{1+2^{-1}} + \frac{2}{1+2^{-2}} + \frac{3}{1+2^{-3}} + \ldots$$

2^{-n}

Here
$$u_n = \frac{n}{1+2^{-n}} = \frac{n}{1+\frac{1}{2^n}}$$

Let
$$u_n = n$$

Let us compare Σu_n with Σv_n

$$\frac{u_n}{v_n} = \frac{n}{1+\frac{1}{2^n}}\cdot\frac{1}{n} = \frac{1}{1+\frac{1}{2^n}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{2^n}} = \frac{1}{1+0} = 1$$

Which is finite and non-zero.

$\therefore$ Σu_n and Σv_n, converge or diverge together since $\Sigma v_n = \Sigma \frac{1}{n}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = 1$.

$\therefore$ Σv_n divergent $\Rightarrow$ Σu_n is also divergent.

Ans.

Example 14. *Examine the convergence of the series* $\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+\ldots$

Solution. Here, we have

$$\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+\ldots$$

Here
$$u_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1} = \frac{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}\right)}{n^3\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]} = \frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{n^{\frac{5}{2}}\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]}$$

Let
$$v_n = \frac{1}{n^{\frac{5}{2}}}$$

Let us compare Σu_n with Σv_n,

$$\frac{u_n}{v_n} = \frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{n^{\frac{5}{2}}\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]} \times \frac{n^{\frac{5}{2}}}{1} = \frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]} = \frac{\sqrt{1+0}-0}{(1-0)^3-0} = 1$$

Which is finite and non-zero.

$\therefore$ Σu_n and Σv_n converge or diverge together since $\Sigma v_n = \Sigma \frac{1}{n^{\frac{5}{2}}}$ is of the form $\Sigma \frac{1}{n^p}$.

where $p = \frac{5}{2} > 1$.

$\therefore$ Σv_n is convergent $\Rightarrow$ Σu_n is convergent.

Ans.

Example 15. *Using comparison test discuss the convergence of* $\Sigma \frac{\sqrt{n}-1}{n^2+1}$

(GTU, June. 2012)

Solution. Here we have $\Sigma \frac{\sqrt{n}-1}{n^2+1}$

$$u_n = \frac{\sqrt{n}-1}{n^2+1} = \frac{n^{\frac{1}{2}}\left[1-\frac{1}{\sqrt{n}}\right]}{n^2\left[1+\frac{1}{n^2}\right]} = \frac{\left[1-\frac{1}{\sqrt{n}}\right]}{n^{3/2}\left[1+\frac{1}{n^2}\right]}$$

Let $v_n = n^{-\frac{3}{2}}$

Let us compare Σu_n with Σv_n

$$\frac{u_n}{v_n} = \frac{n^{\frac{3}{2}}\left[1-\frac{1}{\sqrt{n}}\right]}{n^{\frac{3}{2}}\left[1+\frac{1}{n^2}\right]} = \frac{\left[1-\frac{1}{\sqrt{n}}\right]}{\left[1+\frac{1}{n^2}\right]}$$

$$\Rightarrow \qquad \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1-\frac{1}{\sqrt{n}}}{\left[1+\frac{1}{n^2}\right]} = 1$$

Which is finite and non zero.

Therefore, both the series Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n^{\frac{3}{2}}}$ is of the form $\Sigma \frac{1}{n^p}$, where $p = \frac{3}{2}$. Thus Σv_n is convergent $\Rightarrow$ Σu_n is convergent.

Ans.

Example 16. *Test the convergence and divergence of the following series.*

Sessional) Imp II

$$\sum_{n=1}^{\infty}\frac{2n^2+3n}{5+n^5}$$

(Gujarat, I Semester, Jan. 2009)

Solution. Here,
$$u_n = \frac{2n^2+3n}{5+n^5} = \frac{n^2\left(2+\frac{3}{n}\right)}{n^5\left(\frac{5}{n^5}+1\right)} = \frac{1}{n^3}\,\frac{2+\frac{3}{n}}{\frac{5}{n^5}+1}$$

Let
$$v_n = \frac{1}{n^3}$$

By Comparison Test

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n^3\left(2+\frac{3}{n}\right)}{n^3\left(\frac{5}{n^5}+1\right)} = \lim_{n\to\infty}\frac{2+\frac{3}{n}}{\frac{5}{n^5}+1} = 2 = \text{ Finite number.}$$

According to comparison test both series converge or diverge together but Σv_n is convergent as $p = 2$.

Ans.

Hence, the given series is convergent.

Example 17. *Test the following series for convergence* $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots$

Solution. Given series is $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots$

Here
$$u_n=\frac{n+1}{n^p}=\frac{1+\frac{1}{n}}{n^{p-1}}$$

$$v_n=\frac{1}{n^{p-1}} \qquad \therefore \frac{u_n}{v_n}=\frac{1+\frac{1}{n}}{n^{p-1}}\times\frac{n^{p-1}}{1}=1+\frac{1}{n} \quad \lim_{n\to\infty}\frac{u_n}{v_n}=1$$

Therefore, both the series are either convergent or divergent.
But Σv_n is convergent if $p-1>1$, *i.e.*, if $p>2$
and is divergent if $p-1\le 1$, *i.e.*, if $p\le 2$
The given series is convergent if $p>2$ and divergent if $p\le 2$. (P series) Ans.

Example 18. *Using comparison test, discuss the convergence of* $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{1}{n}\right)$ *(GTU, June 2012)*

Solution. Here we have
$$u_n=\frac{1}{n}\sin\frac{1}{n}$$

Since $\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}=1$, it follows that $\sin\frac{1}{n}\sim\frac{1}{n}$ and so $u_n\sim\frac{1}{n^2}$.

We therefore, take $v_n=\frac{1}{n^2}$ and then

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{n^2}{n}\sin\frac{1}{n}=\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}=1=\text{finite}$$

Therefore, Σu_n and Σv_n converge or diverge together. But $\Sigma v_n=\Sigma\frac{1}{n^2}$ converges.
Hence, Σu_n is also convergent by comparison test. Ans.

Example 19. *Determine convergence or divergence of series* $\sum_{n=1}^{\infty}\frac{(2n^2-1)^{\frac{1}{3}}}{(3n^3+2n+5)^{\frac{1}{4}}}$ *(GTU, Dec. 2013)*

Solution. Here, we have

$$\sum_{n=1}^{\infty}\frac{(2n^2-1)^{\frac{1}{3}}}{(3n^3+2n+5)^{\frac{1}{4}}}$$

$$u_n=\frac{(2n^2-1)^{\frac{1}{3}}}{(3n^3+2n+5)^{\frac{1}{4}}}$$
$$=\frac{n^{\frac{2}{3}}\left(2-\frac{1}{n^2}\right)^{\frac{1}{3}}}{n^{\frac{3}{4}}\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}$$
$$=\frac{\left(2-\frac{1}{n^2}\right)^{\frac{1}{3}}}{n^{\frac{3}{4}-\frac{2}{3}}\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}=\frac{\left(2-\frac{1}{n^2}\right)^{\frac{1}{3}}}{n^{\frac{1}{12}}\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}$$

Let us compare Σu_n and Σv_n, where

$$v_n=\frac{1}{n^{\frac{1}{12}}}$$

$$\frac{u_n}{v_n}=\frac{n^{\frac{1}{12}}\left(2-\frac{1}{n^2}\right)^{\frac{1}{3}}}{n^{\frac{1}{12}}\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}$$
$$=\frac{\left(2-\frac{1}{n^2}\right)^{\frac{1}{3}}}{\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\left(2-\frac{1}{n^2}\right)^{\frac{1}{3}}}{\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}$$
$$=\frac{2^{\frac{1}{3}}}{3^{\frac{1}{4}}}\neq 0.$$

Which is finite and non zero.

By comparison test Σu_n and Σv_n converge or diverge together, since $\Sigma v_n=\Sigma\frac{1}{n^{\frac{1}{12}}}$ is of the form $\frac{1}{n^p}$

$$P=\frac{1}{12}<1$$

Σv_n is divergent $\Rightarrow \Sigma u_n$ is also divergent. Ans.

EXERCISE 9.5

Examine the Convergence or Divergence of the Following Series

1. $2+\frac{3}{2}\cdot\frac{1}{4}+\frac{4}{3}\cdot\frac{1}{4^2}+\frac{5}{4}\cdot\frac{1}{4^3}+\ldots\infty$ **Ans.** Convergent
2. $1+\frac{1\cdot 2}{1\cdot 3}+\frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5}+\frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7}+\ldots\infty$ **Ans.** Convergent
3. $\frac{1}{1\cdot 2}+\frac{2}{3\cdot 4}+\frac{3}{5\cdot 6}+\ldots\infty$ **Ans.** Divergent
4. $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\ldots\ldots\ldots\infty$ **Ans.** Convergent
5. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\ldots\infty$ **Ans.** Convergent
6. $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\ldots$ **Ans.** Convergent
7. $\frac{1}{3}+\frac{2!}{3^2}+\frac{3!}{3^3}+\ldots\infty$ **Ans.** Convergent
8. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}$ **Ans.** Divergent
9. $\sum_{n=1}^{\infty}\frac{2n^3+5}{4n^5+1}$ **Ans.** Convergent
10. $\sum_{n=1}^{\infty}\frac{a^n}{x^n+n^a}$ **Ans.** If $x > a$, convergent; if $x \le a$, Divergent
11. $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$ **Ans.** Convergent
12. $\sum_{n=1}^{\infty}\sqrt{(n^2+1)}-n$ **Ans.** Divergent
13. $\sum_{n=1}^{\infty}\left[\sqrt{(n^4+1)}-\sqrt{(n^4-1)}\right]$ **Ans.** Convergent
14. $\sum_{n=1}^{\infty}\frac{2^n+1}{3^n+n}$ **Ans.** Convergent
15. $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ **Ans.** Convergent
16. $\sum_{n=1}^{\infty}\frac{n^2}{e^n}$ **Ans.** Convergent

9.18 D'ALEMBERT'S RATIO TEST

Statement. *If Σu_n is a positive term series such that $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=k$ then*

(i) the series is convergent if $k<1$ *(ii) the series is divergent if $k>1$*

Proof.

Case 1. When $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=k<1$

By definition of a limit, we can find a number $r\,(<1)$ such that

$$\frac{u_{n+1}}{u_n}<r \text{ for all } n\ge m \qquad \left[\frac{u_2}{u_1}<r,\frac{u_3}{u_2}<r,\frac{u_4}{u_3}<r\ldots\right]$$

Omitting the first m terms, let the series be

$$u_1+u_2+u_3+u_4+\ldots\infty$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\frac{u_4}{u_1}+\ldots\right)=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\ldots\infty\right)$$

$$<u_1(1+r+r^2+r^3+\ldots\infty) \qquad (r<1)$$

$$=\frac{u_1}{1-r}, \text{ which is a finite quantity.}$$

Hence, Σu_n is convergent.

K<1

Case 2. When $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=k>1$

K>1

By definition of limit, we can find a number m such that $\frac{u_{n+1}}{u_n}\ge 1$ for all $n\ge m$

$$\frac{u_2}{u_1}>1, \qquad \frac{u_3}{u_2}>1, \qquad \frac{u_4}{u_3}>1$$

Ignoring the first m terms, let the series be

$$u_1+u_2+u_3+u_4+\ldots\infty$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\frac{u_4}{u_1}+\ldots\right)=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\ldots\infty\right)$$

$$\ge u_1(1+1+1+1\ldots\text{ to } n \text{ terms})=nu_1$$

$$[\because \lim_{n\to\infty}(u_1+u_2+\ldots+u_n)=nu_1]$$

$$\lim_{n\to\infty}S_n\ge\lim_{n\to\infty}nu_1=\infty$$

$\frac{1}{n}$

$\frac{U_{n+1}}{U_n}$

Hence, Σu_n is divergent.

Note: When $\frac{u_{n+1}}{u_n}=1\ (k=1)$

The ratio test fails.

For Example. Consider the series whose n^{th} term $= \dfrac{1}{n}$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1 \quad \ldots(1)$$

Consider the second series whose n^{th} term is $\dfrac{1}{n^2}$.

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = 1 \quad \ldots(2)$$

Thus, from (1) and (2) in both cases $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n} = 1$

But we know that the first series is divergent as $p = 1$.
The second series is convergent as $p = 2$.

Hence, when $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n} = 1$, the series may be convergent or divergent.

Thus, ratio test fails when $k = 1$. **Ans.**

Example 20. *Prove that* $1+\dfrac{2}{3}+\dfrac{4}{9}+\dfrac{8}{27}+\dfrac{16}{81}+\ldots$ *converges and find it's sum.*

V Imp (Sem)

(G.T.U. Dec. 2014, 2011)

Solution. Here we have $1+\dfrac{2}{3}+\dfrac{4}{9}+\dfrac{8}{27}+\dfrac{16}{81}+\ldots$

$$u_n = \left(\frac{2}{3}\right)^{n-1}$$

$$u_{n+1} = \left(\frac{2}{3}\right)^{n}$$

$$\frac{u_{n+1}}{u_n} = \left(\frac{2}{3}\right)^{n}\left(\frac{3}{2}\right)^{n-1}$$

By D'Alemberts Test

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{2}{3}$$

$$= \frac{2}{3} < 1$$

R < 1
Con
R > 1

Hence Σu_n is convergent. **Ans.**

Example 21. *Test for convergence of the series whose n^{th} term is* $\dfrac{n^2}{2^n}$.

Solution. Here, we have $u_n = \dfrac{n^2}{2^n}$, $\quad u_{n+1} = \dfrac{(n+1)^2}{2^{n+1}}$

By D'Alembert's Test

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2} = \lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right)^2 = \frac{1}{2} < 1$$

Hence, the series is convergent by D'Alembert's Ratio Test. **Ans.**

Example 22. *Test for convergence the series whose n^{th} term is* $\dfrac{2^n}{n^3}$.

Solution. Here, we have $u_n = \dfrac{2^n}{n^3}$, $\quad u_{n+1} = \dfrac{2^{n+1}}{(n+1)^3}$

By D'Alembert's Ratio Test

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}}{(n+1)^3}\cdot\frac{n^3}{2^n} = \frac{2}{\left(1+\frac{1}{n}\right)^3} \Rightarrow \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{2}{\left(1+\frac{1}{n}\right)^3} = 2 > 1$$

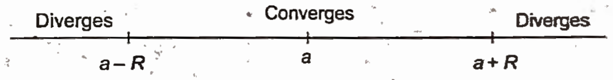
Hence, the series is divergent.

Interval of Convergence and Radius of Convergence

Interval of convergence of a power series is the interval of x say $-a < x < a$ such that the series converges for the value of x in the interval $(-a, a)$ and diverges for the values of x outside the interval.

Radius of Convergence

Radius of convergence is the half length of the interval for example the series converges for all x some finite open interval $(a - R, a + R)$ and diverges.
If x is less than $a - R$ or $x > a + R$



Here were the interval $(a - R, a + R)$

Radius of convergence for the series is R and 'a' is the centre. Radius of convergence is the radius of biggest circle in which series converges.

Example 23. *Find the radius of convergence for the series* $\sum\limits_{n=1}\dfrac{x^n}{n+2}$ (GTU, March, 2009)

Solution. We have $\quad u_n = \dfrac{x^n}{n+2}$

$$\Rightarrow \qquad u_{n+1} = \frac{x^{n+1}}{n+3}$$

$$\therefore \qquad \frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{n+3}\cdot\frac{n+2}{x^n} = \left(\frac{n+2}{n+3}\right)x = \left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)x$$

$$\Rightarrow \qquad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = |x|$$

$\therefore$ By ratio test, the series converges if $|x| < 1$ and diverges if $|x| > 1$.
Hence, the radius of convergence R is 1. **Ans.**

Example 24. *Test the convergence of the series:*

$$\sqrt{\frac{1}{2}}x+\sqrt{\frac{2}{5}}x^2+\sqrt{\frac{3}{10}}x^3+\ldots\infty,\ x>0$$

(GTU, June 2012)

Solution. Here, we have

$$u_n=\sqrt{\frac{n}{n^2+1}}x^n$$

$$u_{n+1}=\sqrt{\frac{n+1}{(n+1)^2+1}}\cdot x^{n+1}$$

$$\frac{u_{n+1}}{u_n}=\frac{\sqrt{n+1}}{\sqrt{n^2+2n+2}}x^{n+1}\times\frac{\sqrt{n^2+1}}{\sqrt{n}}\cdot\frac{1}{x^n}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}}\sqrt{1+\frac{1}{n^2}}}{\sqrt{1+\frac{2}{n}+\frac{2}{n^2}}}\cdot x=x$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n converges if $x<1$ and diverges if $x>1$.

If $x=1$, test fails.

When $x=1$, the Ratio Test fails.

When $x=1$, $\quad u_n=\sqrt{\frac{n}{n^2+1}}=\sqrt{\frac{n}{n^2\left(1+\frac{1}{n^2}\right)}}=\frac{1}{\sqrt{n}}\cdot\frac{1}{\sqrt{1+\frac{1}{n^2}}}$

$$A_n=\frac{1}{\sqrt{n}}$$

$$\frac{u_n}{v_n}=\frac{1}{\sqrt{n}}\cdot\frac{1}{\sqrt{1+\frac{1}{n^2}}}\cdot\frac{\sqrt{n}}{1}=\frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}}=1$$

Which is finite and non-zero.

$\therefore$ By comparison test, Σu_n and Σv_n converge or diverge together.

Since $\sum v_n=\sum\frac{1}{\sqrt{n}}$ is of the form $\sum\frac{1}{n^p}$ with $p=\frac{1}{2}<1$.

$\sum v_n$ diverges $\Rightarrow\sum u_n$ diverges.

Hence, the given series Σu_n converges as $x<1$ and diverges if $x\geq 1$.

Here radius of convergence is 1. **Ans.**

Example 25. *Determine absolute or conditional convergence of the series* $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{n^2}{n^3+1}$.

[G.T.U, Dec. 2013]

Solution. Here, we have

$$u_n=(-1)^n\frac{n^2}{n^3+1}$$

$$u_{n+1}=(-1)^{n+1}\frac{(n+1)^2}{(n+1)^3+1}$$

By D'Alemberts Test

$$\frac{u_{n+1}}{u_n}=\frac{(-1)^{n+1}(n+1)^2}{(n+1)^3+1}\times\frac{(n^3+1)}{(-1)^n n^2}$$

$$=-\frac{(n^2+2n+1)(n^3+1)}{(n^3+3n^2+3n+1+1)n^2}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}-\frac{(n^2+2n+1)(n^3+1)}{(n^3+3n^2+3n+2)n^2}$$

$$=\lim_{n\to\infty}-\frac{\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1+\frac{1}{n^3}\right)}{\left(1+\frac{3}{n}+\frac{3}{n^2}+\frac{2}{n^3}\right)\cdot 1}=-1$$

D'Alembert's Ratio Test, the series is divergent. **Ans.**

Example 26. *Investigate the convergence of the series* $\sum_{n=1}^{\infty}\frac{2^n+5}{3^n}$. (GTU, June 2015)

Solution. Here, we have

$\frac{U_{n+1}}{U_n}$

$$u_n=\frac{2^n+5}{3^n}$$

$$u_{n+1}=\frac{2^{n+1}+5}{(3)^{n+1}}$$

$$\frac{u_{n+1}}{u_n}=\frac{2^{n+1}+5}{3^{n+1}}\times\frac{3^n}{2^n+5}$$

By D'Alemberts Test

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{2^{n+1}+5\cdot 3^n}{3^{n+1}(2^n+5)}=\lim_{n\to\infty}\frac{2^{n+1}+5}{3\cdot(2^n+5)}=\lim_{n\to\infty}\frac{2+\frac{5}{2n}}{3\left(1+\frac{5}{2n}\right)}$$

$$=\frac{2}{3}<1$$

Hence by *D'Alemberts Ratio Test* Σu_n is convergent. **Ans.**

Example 27. *Find value of x for which the given series* $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+...$

(GTU, Dec., 2013)

Solution. Here, we have $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+...$

$$u_n=\frac{1}{(n+1)\sqrt{n}}x^{2(n-1)}$$

$$u_{n+1}=\frac{1}{(n+2)\sqrt{n+1}}x^{2n}$$

By D'Alemberts Ratio Test

$$\frac{u_{n+1}}{u_n}=\left[\frac{1}{(n+2)}\frac{x^{2n}}{\sqrt{n+1}}\times\frac{(n+1)\sqrt{n}}{x^{2n-2}}\right]$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{\sqrt{n+1}\sqrt{n}}{(n+2)}x^2$$

$$=\lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}}}{1+\frac{2}{n}}x^2$$

$$=x^2$$

(*i*) If $x^2<1$, then Σu_n is convergent.

(*ii*) If $x^2>1$, then Σu_n is divergent.

(*iii*) If $x=1$, then D'Almberts test fails

By comparison Test

$$u_n=\frac{1}{(n+1)\sqrt{n}}$$

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{1}{n^{\frac{3}{2}}\left(1+\frac{1}{n}\right)}$$

Let

$$v_n=\frac{1}{n^{3/2}}$$

$$\frac{u_n}{v_n}=\frac{n^{3/2}}{n^{3/2}\left(1+\frac{1}{n}\right)}$$

$$\lim\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)}$$

$$=1=\text{finite}$$

Which is finite and non-zero.

By comparison test Σu_n and Σv_n converge or diverge together.

Since $v_n=\frac{1}{n^{3/2}}$ which is of the form $\Sigma\frac{1}{n^p}$ with $P=\frac{3}{2}>1$

Σv_n converges $\Rightarrow \Sigma u_n$ converges.

Hence, the given series Σu_n converges as $n\le 1$ and diverge if $x>1$. **Ans.**

Example 28. *Find the interval of convergence of the series* $\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}}$ (GTU, Dec. 2013)

Solution. Here, we have the series $\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}}$

$$u_n=\frac{(-3)^n x^n}{\sqrt{n+1}}$$

$$u_{n+1}=\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}$$

$$\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}}{\frac{(-3)^n x^n}{\sqrt{n+1}}}\right|$$

$$=\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\times\frac{\sqrt{n+1}}{(-3)^n x^n}\right|$$

$$=\left|\frac{-3x\sqrt{n+1}}{\sqrt{n+2}}\right|$$

$$\left|\lim_{n\to\infty}\frac{u_{n+1}}{u_n}\right|=\left|-3x\sqrt{\frac{n+1}{n+2}}\right|$$

$$=3\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}}|x|$$

$$=3|x|$$

By D'Alembert's Ratio Test

(*i*) Σu_n is convergent if $3|x|<1$

(*ii*) Σu_n is divergent if $|x|>\frac{1}{3}$

Radius of convergence is $R=\frac{1}{3}$

The series converges in the interval $\left(-\frac{1}{3},\frac{1}{3}\right)$ if $x=\frac{1}{3}$ then

$$\sum_{n=0}^{\infty}\frac{(-3)^n\left(-\frac{1}{3}\right)^n}{\sqrt{n+1}}=\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}=\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots..$$

Here $\qquad P=\frac{1}{2}<1$

Hence Σu_n is divergent.

Example 29. *Investigate the convergence of the series* $\sum_{n=1}^{\infty}\frac{1}{1+2^2+3^2+\ldots+n^2}$

(GTU, Dec. 2015)

Solution. Here, we have

$$u_n=\frac{1}{1+2^2+3^2+\ldots+n^2}$$

$$=\frac{1}{\frac{n(n+1)(2n+1)}{6}}$$

$$=\frac{6}{n(n+1)(2n+1)}$$

$$u_{n+1}=\frac{6}{(n+1)(n+2)(2n+3)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{6}{(n+1)(n+2)(2n+3)}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$=\lim_{n\to\infty}\frac{n(2n+1)}{(n+2)(2n+3)}$$

$$=\lim_{n\to\infty}\frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}=\frac{2}{2}=1$$

By D'Alembert's Ratio test fails

Now we apply Raabe's test

$$\lim_{n\to\infty}n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty}n\left(\frac{(n+2)(2n+3)}{n(2n+1)}-1\right)$$

$$=\lim_{n\to\infty}n\left[\frac{2n^2+7n+6-2n^2-n}{n(2n+1)}\right]$$

$$=\lim_{n\to\infty}n\left(\frac{6n+6}{n(n+1)}\right)=\lim_{n\to\infty}\frac{6n+6}{2n+1}$$

$$=\lim_{n\to\infty}\frac{n\left(6+\frac{6}{n}\right)}{n\left(2+\frac{1}{n}\right)}=\lim_{n\to\infty}\frac{6+\frac{6}{n}}{2+\frac{1}{n}}$$

$$=3>1$$

Hence the series is convergent.

Example 30. *For the series* $\sum_{n=1}^{\infty}\frac{(-1)^n(x+2)^n}{n}$ *find the series' radius and interval of convergence. For what value of x does the series converge absolutely, conditionally?*

(GTU, Dec., 2015)

Solution. Here, we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n(x+2)^n}{n}$$

$$u_n=\frac{(-1)^n(x+2)^n}{n}$$

$$u_{n+1}=\frac{(-1)^{n+1}(x+2)^{n+1}}{n+1}$$

By D'Alemberts Ratio Test

$$\frac{u_{n+1}}{u_n}=\frac{(-1)^{n+1}(x+2)^{n+1}}{n+1}\times\frac{n}{(-1)^n(x+2)^n}$$

$$=-(x+2)\frac{n}{n+1}$$

$$=-(x+2)\frac{1}{1+\frac{1}{n}}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left[-(x+2)\frac{1}{1+\frac{1}{n}}\right]$$

$$=-(x+2)$$

If $-(x+2)<1\Rightarrow -3<x$ then Σu_n is convergent.

If $-(x+2)>1\Rightarrow -3>x$, then Σu_n is divergent.

If $-(x+2)=1\Rightarrow x=-3$ then D'Alemberts Test fails.

$$\Sigma u_n=\frac{(-1)^n(-3+2)^n}{n}=\frac{(-1)^n(-1)^n}{n}=\frac{1}{n}$$

$$=\frac{1}{n}\text{ is of the form }=\frac{1}{n^P}\text{ with }P=1$$

Σu_n is divergent

Σu_n is convergent $-3<x$ and divergent $-3\le x$ **Ans.**

EXERCISE 9.6

Test the convergence for series:

1. $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ **Ans.** Convergent
2. $\sum_{n=1}^{\infty} \frac{n!}{n^2}$ **Ans.** Divergent
3. $\left(\frac{1}{3}\right)^2+\left(\frac{1\cdot2}{3\cdot5}\right)^2+\left(\frac{1\cdot2\cdot3}{3\cdot5\cdot7}\right)^2+\ldots\infty$ **Ans.** Convergent
4. $\frac{2}{1}+\frac{2\cdot5\cdot8}{1\cdot5\cdot9}+\frac{2\cdot5\cdot8\cdot11}{1\cdot5\cdot9\cdot13}+\ldots\infty$ **Ans.** Convergent
5. $\sum_{n=1}^{\infty} \frac{n!\,2^n}{n^n}$ **Ans.** Convergent
6. $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n\cdot 3^n}$ **Ans.** Convergent if $x > 3$, Divergent if $x<3$
7. Prove that, if $u_{n+1}=\frac{k}{1+u_n}$, where $k>0$, $u_1>0$, then the series Σu_n converges to the positive root of the equation $x^2+x=k$.

9.19 RAABE'S TEST (HIGHER RATIO TEST)

$\frac{U_4}{U_n}$

If Σu_n is a positive term series such that $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=k$, then

(*i*) the series is convergent if $k>1$ (*ii*) the series is divergent if $k<1$.

Proof. Case I. $k>1$

Let p be such that $k>p>1$ and compare the given series Σu_n with $\sum\frac{1}{n^p}$ which is convergent as $p>1$.

$$\frac{u_n}{u_{n+1}}>\frac{(n+1)^p}{n^p} \quad \text{or} \quad \left(\frac{u_n}{u_{n+1}}\right)>\left(1+\frac{1}{n}\right)^p>1+\frac{p}{n}+\frac{p(p-1)}{2!}\frac{1}{n^2}+\ldots$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right)>p+\frac{p(p-1)}{2!}\frac{1}{n}+\ldots \qquad \text{(Binomial Theorem)}$$

If

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)>p$$

and $k>p$ which is true as $k>p>1$; Σu_n is convergent when $k>1$.

Case II. $k<1$ Same steps as in Case I.

Notes:

1. Raabe's Test fails if $k=1$
2. Raabe's Test is applied only when D'Alembert's Ratio Test fails.

Example 31. *Is the series* $\sum_{n=1}^{\infty}\frac{2n+1}{(n+1)^2}$ *converges or diverges.* (G.T.U. June 2015)

Solution. Here, we have

$$\sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}\frac{2n+1}{(n+1)^2}$$

$$u_n=\frac{2n+1}{(n+1)^2}$$

$$u_{n+1}=\frac{2n+3}{(n+2)^2}$$

By D'Alembert's Test Ratio

$$\frac{u_{n+1}}{u_n}=\frac{(2n+3)}{(n+2)^2}\times\frac{(n+1)^2}{2n+1}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(2n+3)}{(2n+1)}\frac{(n+1)^2}{(n+2)^2}$$

$$=\lim_{n\to\infty}\frac{\left(2+\frac{3}{n}\right)\left(1+\frac{1}{n}\right)^2}{\left(2+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^2}$$

$$=1$$

$\frac{u_{n+1}}{u_n}$

D'Alembert's ratio test fails.

$n\left(\frac{u_n}{u_{n+1}}-1\right)$

By Raabe's Test

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right]=\lim_{n\to\infty} n\left[\frac{(2n+1)(n+2)^2}{(2n+3)(n+1)^2}-1\right]$$

$$=\lim_{n\to\infty} n\left[\frac{2n^3+9n^2+12n+4}{2n^3+7n^2+8n+3}-1\right]$$

$$=\lim_{n\to\infty} n\left[\frac{2n^3+9n^2+12n+4-2n^3-7n^2-8n-3}{2n^3+7n^2+8n+3}\right]$$

$$=\lim_{n\to\infty} n\left[\frac{2n^2+4n+1}{2n^3+7n^2+8n+3}\right]$$

$$=\lim_{n\to\infty}\left[\frac{2n^3+4n^2+n}{2n^3+7n^2+8n+3}\right]$$

$$=\lim_{n\to\infty}\left[\frac{2+\frac{4}{n}+\frac{1}{n^2}}{2+\frac{7}{n}+\frac{8}{n^2}+\frac{3}{n^3}}\right]$$

$$=1$$

Raabe's test fails.

By Gauss's Test

$$\frac{u_n}{u_{n+1}} = \frac{(n+2)^2(2n+1)}{(2n+3)(n+1)^2}$$

$$= \frac{2n^3+9n^2+12n+4}{2n^3+7n^2+8n+3}$$

$$= 1+\frac{1}{n}+\frac{-\frac{3}{2}}{n^2}$$

$$= \alpha+\frac{\beta}{n}+\frac{\gamma}{n^2}$$

$\alpha = 1, \beta = 1$

$\sum_{n=1}^{\infty} u_n$ is divergent **Ans.**

Example 32. *Test for convergence the series* $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+.....$ (G.T.U., June 2014)

Solution. Here, we have

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+.....$$

$$u_n = \frac{1}{n(n+1)}$$

$$u_{n+1} = \frac{1}{(n+1)(n+2)}$$

By D'Alembert Ratio Test

$$\frac{u_{n+1}}{u_n} = \frac{n(n+1)}{(n+1)(n+2)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \frac{n}{n+2}$$

$$= \frac{1}{1+\frac{2}{n}}$$

$$= 1$$

D'Alembert's Ratio test fails.

By Raabe's Test

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{n+2}{n}-1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{n+2-n}{n}\right)$$

$$= 2$$

$$= 2 > 1$$

Hence Σu_n is convergent. **Ans.**

V.Imp

Example 33. *Test the convergence for the series* $\frac{x}{1\cdot 2}+\frac{x^2}{3\cdot 4}+\frac{x^3}{5\cdot 6}+\frac{x^4}{7\cdot 8}+...$ (SM)

(MTU. 2014, M.U. 2009)

Solution. Here, $u_n = \frac{x^n}{(2n-1)2n}$ and $u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$

By D'Alembert's Test

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(2n+1)(2n+2)} \times \frac{(2n-1)2n}{x^n} = \lim_{n\to\infty}\frac{x\left(1-\frac{1}{2n}\right)}{\left(1+\frac{1}{2n}\right)\left(1+\frac{2}{2n}\right)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = x$$

(*i*) If $x < 1$, Σu_n is convergent (*ii*) If $x > 1$, Σu_n is divergent (*iii*) If $x = 1$, Test fails.

Let us apply **Raabe's Test** when $x = 1$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(2n+1)(2n+2)}{2n(2n-1)}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(2n+1)(2n+2)-2n(2n-1)}{2n(2n-1)}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(8n+2)}{2n(2n-1)}\right] = \lim_{n\to\infty}\frac{\left(2+\frac{2}{4n}\right)}{1\left(1-\frac{1}{2n}\right)} = 2 > 1$$

The series is convergent.

Hence we can say that the given series is convergent if $x \le 1$ and divergent, if $x > 1$.

Here, the radius of convergence is 1. **Ans.**

Example 34. *Test the following series for convergence* $\sum\frac{1}{\sqrt{n+1}-1}$

Solution. Here, $u_n = \frac{1}{\sqrt{n+1}-1}, u_{n+1} = \frac{1}{\sqrt{n+2}-1}$

$$\frac{u_{n+1}}{u_n} = \frac{\sqrt{n+1}-1}{\sqrt{n+2}-1}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}-\frac{1}{n^2}}}{\sqrt{1+\frac{2}{n}-\frac{1}{n^2}}} = 1$$

D'Alembert's test fails.

By Raabe's Test

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{\sqrt{n+2}-1}{\sqrt{n+1}-1}-1\right)$$

$$= \lim_{n\to\infty} n\left[\frac{\sqrt{n+2}-1-\sqrt{n+1}+1}{\sqrt{n+1}-1}\right] = \lim_{n\to\infty} n\left[\frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{n+1}-1}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\sqrt{1+\frac{2}{n}}-\sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{1}{n}-\frac{1}{n^{1/2}}}}\right] = 0 < 1$$

Hence, Σu_n is divergent. **Ans.**

Example 35. *Discuss the convergence of the series:*

$$\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\ldots(x>0)$$

Solution. Here, we have

$$\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\ldots$$

Neglecting the first term, we have

$$u_n = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}\cdot\frac{x^{2n+1}}{2n+1}$$

and $$u_{n+1} = \frac{1\cdot 3\cdot 5\ldots(2n-1)(2n+1)}{2\cdot 4\cdot 6\ldots(2n)(2n+2)}\cdot\frac{x^{2n+3}}{2n+3}$$

By D'Alembert's Test

$$\frac{u_{n+1}}{u_n} = \frac{2,4,6\ldots(2n)(2x+1)}{1\cdot 3\cdot 5\ldots(2n-1)x^{2n+1}}\times\frac{1,3\cdot 5\ldots(2n-1)(2n+1)x^{2n+3}}{2\cdot 4\cdot 6\ldots(2n)(2n+2)(2n+3)}$$

$$= \frac{(2n+1)(2n+1)}{(2n+2)(2n+3)}x^2$$

$$= \frac{\left(2+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{\left(2+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}x^2$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\left(2+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{\left(2+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}x^2$$

$$= x^2$$

If $x^2 < 1$, Σu_n is convergent

If $x^2 > 1$, Σu_n is divergent

If $x^2 = 1$, Test fails.

Now Raabe's test

When $x^2 = 1$, we have $\dfrac{u_n}{u_{n+1}} = \dfrac{(2n+2)(2n+3)}{(2n+1)^2} = \dfrac{4n^2+10n+6}{4n^2+4n+1}$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+4n+1}-1\right)$$

$$= \lim_{n\to\infty}\frac{6n^2+5n}{4n^2+4n+1} = \lim_{n\to\infty}\frac{6+\frac{5}{n}}{4+\frac{4}{n}+\frac{1}{n^2}} = \frac{6}{4} = \frac{3}{2} > 1$$

$\therefore$ By Raabe's Test, the series converges.

Hence, Σu_n is convergent if $x^2 \le 1$ and divergent if $x^2 > 1$. **Ans.**

Example 36. *Test the convergence of* $\dfrac{1}{1\cdot 2\cdot 3}+\dfrac{3}{2\cdot 3\cdot 4}+\dfrac{5}{3\cdot 4\cdot 5}+\ldots$ *(G.T.U, Dec. 2014)*

Solution. Here, we have

$$\frac{1}{1\cdot 2\cdot 3}+\frac{3}{2\cdot 3\cdot 4}+\frac{5}{3\cdot 4\cdot 5}+\ldots$$

$$u_n = \frac{1}{n(n+1)(n+2)}$$

$$u_{n+1} = \frac{1}{(n+1)(n+2)(n+3)}$$

By D'Alemberts Ratio Test

$$\frac{u_{n+1}}{u_n} = \frac{1}{(n+1)(n+2)(n+3)}\times n(n+1)(n+2)$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{n}{n+3}$$

$$= \lim_{n\to\infty}\frac{1}{1+\frac{3}{n}}$$

$$= 1$$

D'Alemberts ratio test fails.

By Raabe's Test

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{n+3}{n}-1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{n+3-n}{n}\right) = 3$$

$$= 3 > 1$$

Hence Σu_n is convergent by Raabe's test. **Ans.**

D.Imp **Example 37.** *Test the convergence of the series.* $\frac{1\cdot 2}{3^2\cdot 4^2}+\frac{3\cdot 4}{5^2\cdot 6^2}+\frac{5\cdot 6}{7^2\cdot 8^2}+\ldots$

(G.T.U., Dec. 2013)

Solution. Here, we have

$$\frac{1\cdot 2}{3^2\cdot 4^2}+\frac{3\cdot 4}{5^2\cdot 6^2}+\frac{5\cdot 6}{7^2\cdot 8^2}+\ldots$$

$$u_n=\frac{n(n+1)}{(n+2)^2(n+3)^2}$$

$$u_{n+1}=\frac{(n+1)(n+2)}{(n+3)^2(n+4)^2}$$

By D'Alembert's Ratio Test

$$\frac{u_{n+1}}{u_n}=\frac{(n+1)(n+2)}{(n+3)^2(n+4)^2}\times\frac{(n+2)^2(n+3)^2}{n(n+1)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+2)(n+2)^2}{(n+4)^2 n}$$

$$=\lim_{n\to\infty}\frac{(n+2)^3}{n(n+4)^2}$$

$$=\lim_{n\to\infty}\frac{n^3+6n^2+12n+8}{n^3+8n^2+16n}$$

$$=\lim_{n\to\infty}\frac{1+\frac{6}{n}+\frac{12}{n^2}+\frac{8}{n^3}}{1+\frac{8}{n}+\frac{16}{n^2}}$$

$$=1$$

D'Alembert's ratio test fails

By Raabes Test

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{n^3+8n^2+16n}{n^3+6n^2+12n+8}-1\right)$$

$$=\lim_{n\to\infty} n\left(\frac{n^3+8n^2+16n-n^3-6n^2-12n-8}{n^3+6n^2+12n+8}\right)$$

$$=\lim_{n\to\infty} n\left(\frac{2n^2+4n-8}{n^3+6n^2+12n+8}\right)$$

$$=\lim_{n\to\infty}\frac{2n^3+4n^2-8n}{n^3+6n^2+12n+8}$$

$$=\lim_{n\to\infty}\frac{2+\frac{4}{n}-\frac{8}{n^2}}{1+\frac{6}{n}+\frac{12}{n^2}+\frac{8}{n^3}}$$

$$=2>1$$

Here By Raabe's test Σu_n is convergent. **Ans.**

Example 38. *Test the following series for convergence*

$$\frac{1}{2}x+x^2+\frac{9}{8}x^3+x^4+\frac{25}{32}x^5+\ldots\infty$$

Solution. Here, $\quad u_n=\frac{n^2\cdot x^n}{2^n},\qquad u_{n+1}=\frac{(n+1)^2\cdot x^{n+1}}{2^{n+1}}$

By D'Alembert's Test

$$\frac{u_{n+1}}{u_n}=\frac{(n+1)^2 x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{n^2x^n}=\left(\frac{n+1}{n}\right)^2\frac{x}{2}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\frac{x}{2}=\frac{x}{2}$$

(*i*) If $\frac{x}{2}<1$ or $x<2$, then Σu_n is convergent.

(*ii*) If $\frac{x}{2}>1$ or $x>2$, then Σu_n is divergent.

(*iii*) If $\frac{x}{2}=1$ or $x=2$, then the test fails.

Let us apply **Raabe's test**

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left[\frac{n^2}{(n+1)^2}\frac{2}{2}-1\right]=n\left[\frac{n^2-n^2-2n-1}{(n+1)^2}\right]=\frac{-2n^2-n}{(n+1)^2}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty}\frac{-2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)^2}=-2<1$$

Hence, Σu_n, is divergent if $x\geq 2$, and convergent if $x<2$. **Ans.**

Here the radius of convergence is 2.

Example 39. *Show that the series* $\frac{1}{x}+\frac{2!}{x(x+1)}+\frac{3!}{x(x+1)(x+2)}+\ldots$ *converges if $x>2$ and diverges if $x<2$.*

Solution. Here, $\quad u_n=\frac{n!}{x(x+1)(x+2)\ldots(x+n-1)}$

$$u_{n+1}=\frac{(n+1)!}{x(x+1)(x+2)\ldots(x+n-1)(x+n)}$$

By D'Alembert's Test

$$\frac{u_{n+1}}{u_n}=\frac{n+1}{(x+n)},\qquad \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1+\frac{x}{n}}=1$$

D' Alembert's ratio test fails.

Let us apply **Raabe's Test.**

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{x+n}{n+1}-1\right) = \lim_{n\to\infty} n\left(\frac{x-1}{n+1}\right) = \lim_{n\to\infty}\frac{x-1}{1+\frac{1}{n}} = x-1$$

If $x-1>1$ or $x>2$, then Σu_n is convergent.

Here the radius of convergence is 2.

If $x-1<1$ or $x<2$, then Σu_n is divergent. **Ans.**

Example 40. *Discuss the convergence of the series* $\dfrac{x^2}{2\log 2}+\dfrac{x^3}{3\log 3}+\dfrac{x^4}{4\log 4}+\ldots$

Solution. Here, we have $\dfrac{x^2}{2\log 2}+\dfrac{x^3}{3\log 3}+\dfrac{x^4}{4\log 4}+\ldots$

$$u_n = \frac{x^{n+1}}{(n+1)\log(n+1)}, \qquad u_{n+1} = \frac{x^{n+2}}{(n+2)\log(n+2)}$$

By D'Alembert's Test

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{x^{n+1}}{(n+2)\log(n+2)}\times\frac{(n+1)\log(n+1)}{x^{n+1}}$$

$$= \lim_{n\to\infty} x\left(\frac{n+1}{n+2}\right)\frac{\log(n+2)}{\log(n+2)}$$

$$= \lim_{n\to\infty} x\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)\frac{\log n+\log\left(1+\frac{1}{n}\right)}{\log n+\log\left(1+\frac{2}{n}\right)}$$

$$= \lim_{n\to\infty} x\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)\left[\frac{\log n+\frac{1}{n}-\frac{1}{2}\frac{1}{n^2}+\ldots}{\log n+\frac{2}{n}-\frac{1}{2}\cdot\frac{4}{n^2}+\ldots}\right]$$

$$= \lim_{n\to\infty} x\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)\left[\frac{1+\frac{1}{n\log n}+\ldots}{1+\frac{2}{n\log n}+\ldots}\right] = x$$

(*i*) When $x<1$, the series is convergent

(*ii*) When $x>1$, the series is divergent.

(*iii*) When $x=1$, the test fails.

Let us apply Raabe's test

$$\frac{u_n}{u_{n+1}} = \left(\frac{n+2}{n+1}\right)\frac{\log(n+2)}{\log(n+2)} = \left(\frac{n+2}{n+1}\right)\frac{\log n+\log\left(1+\frac{2}{n}\right)}{\log n+\log\left(1+\frac{1}{n}\right)}$$

$$= \left(\frac{n+2}{n+1}\right)\frac{\log n+\frac{2}{n}-\frac{1}{2}\cdot\frac{4}{n^2}+\ldots}{\log n+\frac{1}{n}-\frac{1}{2}\cdot\frac{1}{n^2}+\ldots} = \left(\frac{n+2}{n+1}\right)\frac{1+\frac{2}{n\log n}+\ldots}{1+\frac{1}{n\log n}+\ldots}$$

$$= \frac{n+2}{n+1}\left(1+\frac{2}{n\log n}\right)\left(1+\frac{1}{n\log n}\right)^{-1} = \frac{n+2}{n+1}\left(1+\frac{2}{n\log n}\right)\left(1-\frac{1}{n\log n}\right)$$

$$= \frac{n+2}{n+1}\left(1+\frac{2}{n\log n}-\frac{1}{n\log n}+\ldots\right) = \left(\frac{n+2}{n+1}\right)\left[1+\frac{1}{n\log n}\right]$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left[\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right]\left[1+\frac{1}{n\log n}\right] = 1+\frac{1}{n\log n}$$

$$n\left[\frac{u_n}{u_{n+1}}-1\right] = n\left[1+\frac{1}{n\log n}-1\right] = \frac{1}{\log n} = 0 < 1$$

Thus the series is divergent when $x = 1$.

Hence, the series converges if $x<1$ and diverges if $x\geq 1$.

Here, the radius of convergence is 1.

Example 41. *Test the series for convergence*

$$1+\frac{\alpha\cdot\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\cdot\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)}x^3+\ldots$$

Solution.
$$u_n = \frac{\alpha(\alpha+1)(\alpha+2)\ldots[\alpha+(n-1)]\cdot\beta(\beta+1)\ldots[\beta+(n-1)]}{n!\ \gamma(\gamma+1)\ldots[\gamma+(n-1)]}x^n$$

$$u_{n+1} = \frac{\alpha(\alpha+1)(\alpha+2)\ldots[\alpha+(n-1)(\alpha+n)\cdot\beta(\beta+1)\ldots[\beta+(n-1)](\beta+n)}{(n+1)!\ \gamma(\gamma+1)\ldots[\gamma+(n-1)](\gamma+n)}x^n$$

By D'Alembert's Test

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}x = \lim_{n\to\infty}\frac{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}\cdot x$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = x$$

(*i*) If $x<1$, the series is convergent.

(*ii*) If $x>1$, the series is divergent.

(*iii*) If $x=1$, the test fails.

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left[\frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}-1\right]=n\left[\frac{n\gamma+n^2+\gamma+n-\alpha\beta-n\alpha-n\beta-n^2}{(\beta+n)(\beta+n)}\right]$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty}\frac{\gamma+\frac{\gamma}{n}+1-\frac{\alpha\beta}{n}-\alpha-\beta}{\left(\frac{\alpha}{n}+1\right)\left(\frac{\beta}{n}+1\right)}=\gamma+1-\alpha-\beta$$

(*i*) If $\gamma+1-\alpha-\beta>1$ or $\gamma>\alpha+\beta$, then Σu_n is convergent.

(*ii*) If $\gamma+1-\alpha-\beta<1$ or $\gamma<\alpha+\beta$, then Σu_n is divergent.

Ans.

Example 42. *Test the Convergence for the series* $\sum_{n=1}^{\infty}\frac{4^n n!n!}{(2n)!}$ *(GTU, Dec 2011, Jan 2011)*

Solution. We have, $u_n=\frac{4^n n!n!}{(2n)!}, u_{n+1}=\frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)!}$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+1)!(n+1)!}{n!n!}\cdot\frac{(2n)!}{(2n+2)!}\cdot\frac{4^{n+1}}{4^n}$$

$$=4\lim_{n\to\infty}\frac{(n+1)(n+1)}{(2n+2)(2n+1)}=\frac{4}{2}\lim_{n\to\infty}\frac{n+1}{2n+1}$$

$$=2\lim_{n\to\infty}\frac{n+1}{2n+1}=2\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2+\frac{1}{n}}$$

$$=2\times\frac{1}{2}=1 \text{ Test fails.}$$

Apply Raabe's ratio test.

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right]=\lim_{n\to\infty} n\left[\frac{n!n!(2n+2)!4^n}{(n+1)!(n+1)!(2n)!4^{n+1}}-1\right]$$

$$=\lim_{n\to\infty} n\left[\frac{2n+1}{2(n+1)}-1\right]=\lim_{n\to\infty} n\left[\frac{2n+1-2n-2}{2(n+1)}\right]$$

$$=\lim_{n\to\infty}\left[\frac{-n}{2n+2}\right]=\lim_{n\to\infty}\left[\frac{-1}{2+\frac{2}{n}}\right]=-\frac{1}{2}<1$$

Hence, the given series is divergent.

Ans.

EXERCISE 9.7

Determine the nature of the following series:

1. $1+\frac{2!}{2^2}+\frac{3!}{3^2}+\frac{4!}{4^2}+...\infty$ **Ans.** Divergent

2. $\frac{1}{1}+\frac{1\cdot3}{1\cdot4}+\frac{1\cdot3\cdot5}{1\cdot4\cdot7}+\frac{1\cdot3\cdot5\cdot7}{1\cdot4\cdot7\cdot10}+...\infty$

   Ans. Convergent

3. $1+\frac{1+\alpha}{1+\beta}+\frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)}+...\infty$ **Ans.** If $\beta-\alpha>1$, convergent. If $\beta-\alpha\le1$, Divergent.

4. $\sum_{n=1}^{\infty}\frac{n^3}{e^n}$

   Ans. Convergent

5. $x+\frac{2x^2}{2!}+\frac{3x^3}{3!}+\frac{4x^4}{4!}+...\infty$

   Ans. Convergent

6. $1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+...$

   Ans. Divergent if $n>1$ & convergent if $x<1$

7. $1+\frac{1}{2}x+\frac{1}{5}x^2+\frac{1}{10}x^3+...$ **Ans.** Divergent if $-1\le x<1$ and divergent if $|x|>1$

8. $1+\frac{(1!)^2}{2!}x^2+\frac{(2!)^2}{4!}x^4+\frac{(3!)^2}{6!}x^6+...\infty (x>0)$

   Ans. Convergent if $x^2<4$, convergent; and divergent if $x^2\ge4$

Find the values of x for which the following series converges:

9. $x^2(\log 2)^q+x^3(\log 3)^q+x^4(\log 4)^q+...$

   Ans. If $x<1$, convergent; and divergent if $x\ge1$

10. $\sum_{n=0}^{\infty}\frac{(-1)^n n!x^n}{10^n}$ diverge value of x.

11. $\sum_{n=0}^{\infty}\frac{x^n}{2n(2n+1)}$ **Ans.** If $x\le1$, convergent; and if $x>1$, divergent

12. $\sum_{n=0}^{\infty}\frac{1\cdot2...n}{4\cdot7...(3n+1)}x^n$ **Ans.** If $0<x<3$, convergent and divergent if $x\ge3$.

13. $1+\frac{(1!)^2}{2!}x+\frac{(2!)^2}{4!}x^2+\frac{(3!)^2}{6!}x^3+.....$ *(M.D.U., Dec. 2010)*

    Ans. convergent if $x<4$; divergent if $x\ge4$.

9.20 GAUSS'S TEST

$\alpha>1$

Statement. *If Σu_n is a positive term series such that*

$$\frac{u_n}{u_{n+1}}=\alpha+\frac{\beta}{n}+\frac{\gamma}{n^2} \qquad \textit{where } \alpha>0$$

(*i*) *if $\alpha>1$, convergent* *if $\alpha<1$, divergent, whatever β may be*

(*ii*) *if $\alpha=1$ and* $\begin{cases}\beta>1, \text{convergent}\\ \beta\le1, \text{divergent}\end{cases}$

$\beta>1$

$|x|>1$

Example 43. *Test for convergence the series* $2+\frac{3}{2}x+\frac{4}{3}x^2+\frac{5}{4}x^3+.....$

(G.T.U., June 2014)

Solution. Here, we have

$$2+\frac{3}{2}x+\frac{4}{3}x^2+\frac{5}{4}x^3+.....$$

$$u_n=\frac{n+1}{n}x^{n-1}$$

$$u_{n+1}=\frac{n+2}{n+1}x^n$$

By D'Alembert's Ratio Test

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left[\frac{n+2}{n+1}x^n\cdot\frac{n}{(n+1)x^{n-1}}\right]$$

$$=\lim_{n\to\infty}\left[\frac{n^2+2n}{n^2+2n+1}x\right]$$

$$=\lim_{n\to\infty}\left[\frac{1+\frac{2}{n}}{1+\frac{2}{n}+\frac{1}{n^2}}\right]x$$

$$=x$$

(*i*) If $x<1$, then Σu_n is convergent.

(*ii*) If $x>1$, then Σu_n is divergent.

(*iii*) If $x=1$, D'Alembert's Ratio Test fails.

Now we apply Raabe's Test

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right]$$

$$=\lim_{n\to\infty} n\left[\frac{n+1}{n}\times\frac{n+1}{n+2}-1\right]$$

$$=\lim_{n\to\infty} n\left[\frac{n^2+2n+1}{n^2+2n}-1\right]$$

$$=\lim_{n\to\infty} n\left[\frac{n^2+2n+1-n^2-2n}{n^2+2n}\right]$$

$$=\lim_{n\to\infty}\left[\frac{n}{n^2+2n}\right]$$

$$=\lim_{n\to\infty}\left(\frac{n}{n^2+2n}\right)$$

$$=\lim_{n\to\infty}\left[\frac{1}{1+\frac{2}{n}}\right]$$

$$=1$$

Raabe's test fails

Let us apply *Gauss test*

$$\frac{u_n}{u_{n+1}}=\alpha+\frac{\beta}{n}+\frac{\gamma}{n^2}$$

$$\frac{\frac{n+1}{n}}{\frac{n+2}{n+1}}=\alpha+\frac{\beta}{n}+\frac{\gamma}{n^2}$$

$$\frac{(n+1)^2}{n(n+2)}=\alpha+\frac{\beta}{n}+\frac{\gamma}{n^2}$$

$$\frac{n^2+2n+1}{n^2+2n}=\alpha+\frac{\beta}{n}+\frac{\gamma}{n^2}$$

$$\begin{array}{r|l} & 1+\frac{1}{n^2} \\ n^2+2n & n^2+2n+1 \\ & n^2+2n \\ & \underline{-\quad -} \\ & 1 \\ & 1+\frac{2}{n} \\ & \underline{-\quad -} \\ & -\frac{2}{n} \end{array}$$

$$\frac{n^2+2n+1}{n^2+2n}=1+\frac{1}{n^2}$$

$$\left(1+\frac{1}{n^2}\right)=\alpha+\frac{\beta}{n}+\frac{\gamma}{n^2}$$

$$\alpha=1,\ \beta=0<1$$

Hence Σu_n is divergent by Gauss Test.

Example 44. *Test for convergence the series*

$$\frac{2^2\cdot 4^2}{3^2\cdot 5^2}+\frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}+\frac{2^2\cdot 4^2\cdot 6^2\cdot 8^2}{3^2\cdot 5^2\cdot 7^2\cdot 9^2}+...$$

Solution. The given series is

$$\frac{2^2\cdot 4^2}{3^2\cdot 5^2}+\frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}+\frac{2^2\cdot 4^2\cdot 6^2\cdot 8^2}{3^2\cdot 5^2\cdot 7^2\cdot 9^2}+...\infty$$

$$u_n = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \ldots (2n+2)^2}{3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \ldots (2n+3)^2}$$

$$u_{n+1} = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \ldots (2n+2)^2 (2n+4)^2}{3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \ldots (2n+3)^2 (2n+5)^2}$$

By D'Alembert's Test

$$\frac{u_{n+1}}{u_n} = \frac{(2n+4)^2}{(2n+5)^2} = \frac{4n^2+16n+16}{4n^2+20n+25} = \frac{4+\frac{16}{n}+\frac{16}{n^2}}{4+\frac{20}{n}+\frac{25}{n^2}}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{4+\frac{16}{n}+\frac{16}{n^2}}{4+\frac{20}{n}+\frac{25}{n^2}} = 1$$

D'Alembert's Test fails. Let us apply **Raabe's Test.**

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+20n+25}{4n^2+16n+16}-1\right)$$

$$= \lim_{n\to\infty}\left(\frac{4n^2+9n}{4n^2+16n+16}\right) = \lim_{n\to\infty}\left[\frac{4+\frac{9}{n}}{4+\frac{16}{n}+\frac{16}{n^2}}\right] = 1, \text{ Raabe's Test fails}$$

Let us apply **Gauss's Test**

$$\frac{u_n}{u_{n+1}} = \frac{(2n+5)^2}{(2n+4)^2} = \frac{\left(1+\frac{5}{2n}\right)^2}{\left(1+\frac{2}{n}\right)^2} = \left(1+\frac{5}{n}+\frac{25}{4n^2}\right)\left(1+\frac{2}{n}\right)^{-2}$$

$$= \left(1+\frac{5}{n}+\frac{25}{4n^2}\right)\left(1-\frac{4}{n}+\frac{(-2)\times(-3)}{2!}\frac{4}{n^2}+\ldots\right) = \left(1+\frac{5}{n}+\frac{25}{4n^2}\right)\left(1-\frac{4}{n}+\frac{12}{n^2}+\ldots\right)$$

$$= 1-\frac{4}{n}+\frac{12}{n^2}+\frac{5}{n}-\frac{20}{n^2}+\frac{25}{4n^2}+\ldots = 1+\frac{1}{n}-\frac{7}{4n^2} \qquad \left(\frac{u_n}{u_{n+1}} = \alpha+\frac{\beta}{n}+\frac{\gamma}{n^2}\right)$$

Hence, $\alpha = 1$, $\beta = 1$. Thus, the series is divergent.

Ans.

9.21 CAUCHY'S INTEGRAL TEST

Statement. *A positive term series $f(1) + f(2) + f(3) + \ldots + f(n)$ where $f(n)$ decreases as n increases, converges or diverges according to the integral*

$$\int_1^\infty f(x)dx$$

is finite or infinite.

Proof. In the figure, the area under the curve from $x = 1$ to $x = n + 1$ lies between the sum of the areas of small rectangles (small height) and sum of the areas of large rectangles (large height).

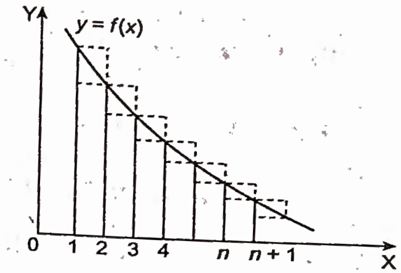
[$f(1), f(2)$ ... represent the height of the rectangles]

$$\Rightarrow \quad f(1)+f(2)+\ldots+f(n) \geq \int_1^{n+1} f(x)dx \geq f(2)+f(3)+\ldots+f(n+1)$$

$$S_n \geq \int_1^{n+1} f(x)dx \geq S_{n+1} - f(1)$$

As $n \to \infty$, from the second inequality that if the integral has a finite value then $\lim_{n\to\infty} S_{n+1}$ is also finite, so $\Sigma f(n)$ is convergent.

Similarly, if the integral is infinite, then from the first inequality that $\lim_{n\to\infty} S_n \to \infty$, so the series is divergent.



Example 45. *Discuss the convergence of integral* $\int_{-2}^{2} \frac{dx}{x^2}$ (G.T.U., Dec. 2013)

Solution. Here, we have

$$\int_{-2}^{2} \frac{dx}{x^2} = -\left[\frac{1}{x}\right]_{-2}^{2} = -\left[\frac{1}{2}+\frac{1}{2}\right] = -\left[\frac{2}{2}\right] = -1$$

Which is finite. Hence it is convergent.

Ans.

Example 46. *Test the convergence of* $\sum_{n=1}^{\infty} \frac{2\tan^{-1} n}{1+n^2}$ (G.T.U. Dec., 2014)

Solution. Here, we have

$$\sum_{n=1}^{\infty} \frac{2\tan^{-1} n}{1+n^2}$$

By Cauchy's Integral test

$$\int_1^\infty f(n)dn = \int_1^\infty \frac{2\tan^{-1} n}{1+n^2} dn$$

$$= \left[(\tan^{-1} n)^2\right]_1^\infty$$

$$= \left[(\tan^{-1} \infty)^2 - (\tan^{-1} 1)^2\right]$$

$$= \left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2$$

$$= \frac{\pi^2}{4} - \frac{\pi^2}{16}$$

$$= \frac{3}{16}\pi^2$$

$= \text{finite}$

tan⁻¹ n = x
1/(1+n²) dn = dx
[(tan⁻¹ n)²]

Hence by Cauchy's Integral Test, Σu_n is convergent.

Ans.

Example 47. *Examine the convergence of* $\sum_{x=2}^{\infty} \frac{1}{x \log x}$.

Solution. Here $$f(x) = \frac{1}{x \log x}$$

$$\int_2^{\infty} \frac{1}{x \log x} dx = \lim_{m \to \infty} [\log \log x]_2^m = \lim_{m \to \infty} [\log \log m - \log \log 2]$$

By Cauchy's Integral Test the series is divergent. **Ans.**

Example 48. *Examine the convergence of* $\sum_{x=1}^{\infty} x e^{-x^2}$ (G.T.U. Dec. 2013)

Solution. Here $$f(x) = x e^{-x^2}$$

Now $$\int_1^{\infty} x e^{-x^2} dx = \lim_{m \to \infty} \left[\frac{e^{-x^2}}{-2}\right]_1^{\infty} = \lim_{m \to \infty} \left[\frac{e^{-\infty^2}}{-2} + \frac{e^{-1}}{2}\right] = \frac{e^{-1}}{2} = \frac{1}{2e},$$ which is finite.

Hence, the given series is convergent. **Ans.**

Example 49. *Check the convergence of* $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$. (G.T.U. Dec. 2013)

Solution. Here, we have

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$
$$= [\sin^{-1} x]_0^1$$
$$= [\sin^{-1}(1) - \sin^{-1}(0)]$$
$$= \frac{\pi}{2} - 0$$
$$= \frac{\pi}{2} \text{ is finite}$$

By Cauchy Integral Test the given series is convergent. **Ans.**

Example 50. *Test the convergence of* $\sum_{n=3}^{\infty} \frac{1}{n \log n \sqrt{\log^2 n - 1}}$ (G.T.U. Dec. 2015)

Solution. Here, we have

$$\sum_{n=3}^{\infty} \frac{1}{n \log n \sqrt{\log^2 n - 1}}$$

Let $$f(n) = \frac{1}{n \log n \sqrt{\log^2 n - 1}}$$

$$\int_3^{\infty} f(n) dn = \int_3^{\infty} \frac{1}{n \log n \sqrt{\log^2 n - 1}} dn$$

$$= \int_{\log_3}^{\infty} \frac{dt}{t\sqrt{t^2-1}} \qquad \left(\text{Let } \log n = t, \frac{1}{n} dn = dt\right)$$

$$= \int_{\log_3}^{\infty} \frac{-\frac{1}{z^2} dz}{\frac{1}{z}\sqrt{\frac{1}{z^2} - 1}} \qquad \left(t = \frac{1}{z}, \quad dt = -\frac{1}{z^2} dz\right)$$

$$= \int_{\log 3}^{\infty} \frac{-dz}{\sqrt{1-z^2}}$$
$$= -(\sin^{-1} z)$$
$$= -\left[\sin^{-1} \frac{1}{t}\right]_{\log 3}^{\infty}$$
$$= -\left[\sin^{-1} \frac{1}{\log n}\right]_{\log 3}^{\infty}$$
$$= -\sin^{-1} \frac{1}{\log \infty} + \sin^{-1} \frac{1}{\log 3}$$
$$= -0 + \text{Finite Quantity}$$
$$= \text{Finite Quantity}$$

Hence by Cauchy Integral test, the given series is convergent. **Ans.**

EXERCISE 9.8

Examine the Convergence:

1. $1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^2}{4^3} + \ldots \infty \; (x > 0)$ **Ans.** Convergent
2. $\frac{2}{1^2} x + \frac{3^2}{2^3} x^2 + \frac{4^2}{3^4} x^3 + \ldots + \frac{(n+1)^n}{n^{n+1}} x^n + \ldots$ **Ans.** $x < 1$, convergent; $x \geq 1$, divergent
3. $1 + \frac{2^2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + \ldots \infty$ **Ans.** Divergent
4. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ **Ans.** Divergent
5. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ **Ans.** Convergent
6. $\sum_{n=1}^{\infty} \frac{1}{n^n}$ **Ans.** Convergent

Example 47. *Examine the convergence of* $\sum_{x=2}^{\infty} \frac{1}{x \log x}$.

Solution. Here
$$f(x) = \frac{1}{x \log x}$$
$$\int_2^{\infty} \frac{1}{x \log x} dx = \lim_{m \to \infty} [\log \log x]_2^m = \lim_{m \to \infty} [\log \log m - \log \log 2]$$

By Cauchy's Integral Test the series is divergent. **Ans.**

Example 48. *Examine the convergence of* $\sum_{x=1}^{\infty} x\, e^{-x^2}$ (G.T.U. Dec. 2013)

Solution. Here
$$f(x) = x\, e^{-x^2}$$

Now $\int_1^{\infty} x\, e^{-x^2} dx = \lim_{m \to \infty} \left[\frac{e^{-x^2}}{-2}\right]_1^{\infty} = \lim_{m \to \infty} \left[\frac{e^{-\infty^2}}{-2} + \frac{e^{-1}}{2}\right] = \frac{e^{-1}}{2} = \frac{1}{2e}$, which is finite.

Hence, the given series is convergent. **Ans.**

Example 49. *Check the convergence of* $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$. (G.T.U. Dec. 2013)

Solution. Here, we have
$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$
$$= \left[\sin^{-1} x\right]_0^1$$
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By Cauchy Integral Test the given series is convergent. **Ans.**

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Solution. Here, we have
$$\sum_{n=3}^{\infty} \frac{1}{n \log n \sqrt{\log^2 n - 1}}$$

Let
$$f(n) = \frac{1}{n \log n \sqrt{\log^2 n - 1}}$$
$$\int_3^{\infty} f(n) dx = \int_3^{\infty} \frac{1}{n \log n \sqrt{\log^2 n - 1}} dn$$

$$= \int_{\log 3}^{\infty} \frac{dt}{t\sqrt{t^2 - 1}} \qquad \left(\text{Let } \log n = t,\ \frac{1}{n} dn = dt\right)$$
$$= \int_{\log 3}^{\infty} \frac{-\frac{1}{z^2} dz}{\frac{1}{z}\sqrt{\frac{1}{z^2} - 1}} \qquad \left(t = \frac{1}{z},\ dt = -\frac{1}{z^2} dz\right)$$
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$$= -(\sin^{-1} z)$$
$$= -\left[\sin^{-1} \frac{1}{t}\right]_{\log 3}^{\infty}$$
$$= -\left[\sin^{-1} \frac{1}{\log n}\right]_{\log 3}^{\infty}$$
$$= -\sin^{-1} \frac{1}{\log \infty} + \sin^{-1} \frac{1}{\log 3}$$
$$= -0 + \text{Finite Quantity}$$
$$= \text{Finite Quantity}$$

Hence by Cauchy Integral test, the given series is convergent. **Ans.**

EXERCISE 9.8

Examine the Convergence:

1. $1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^2}{4^3} + \ldots\infty\ (x > 0)$ **Ans.** Convergent
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3. $1 + \frac{2^2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + \ldots\infty$ **Ans.** Divergent
4. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ **Ans.** Divergent
5. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ **Ans.** Convergent
6. $\sum_{n=1}^{\infty} \frac{1}{n^n}$ **Ans.** Convergent